Open Problems in Number Theoretic Complexity, II

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Introduction.

This conference (ANTS-1) marks the beginning of what we hope will be a long series of international conferences on algorithmic number theory. It seems approppropriate, at the beginning, to state some of the central open problems in the field. Accordingly, this paper contains a list of 36 open problems in number-theoretic complexity. We expect that none of these problems are easy; we are sure that many of them are hard.

This list of problems reflects our own interests and should not be viewed as definitive. As the field changes and becomes deeper, new problems will emerge and old problems will lose favor. Ideally there will be other 'open problems' papers in future ANTS proceedings to help guide the field.

It is likely that some of the problems presented here will remain open for the forseeable future. However, it is possible in some cases to make progress by solving subproblems, or by establishing reductions between problems, or by settling problems under the assumption of one or more well known hypotheses (e.g. the various extended Riemann hypotheses, $\mathcal{NP} \neq \mathcal{P}, \mathcal{NP} \neq \text{co}\mathcal{NP}$).

For the sake of clarity we have often chosen to state a specific version of a problem rather than a general one. For example, questions about the integers modulo a prime often have natural generalizations to arbitrary finite fields, to arbitrary cyclic groups, or to problems with a composite modulus. Questions about the integers often have natural generalizations to the ring of integers in an algebraic number field, and questions about elliptic curves often generalize to arbitrary curves or abelian varieties.

The problems presented here arose from many different places and times. To those whose research has generated these problems or has contributed to our present understanding of them but to whom inadequate acknowledgement is given here, we apologize.

Our list of open problems is derived from an earlier 'open problems' paper we wrote in 1986 [AM86]. When we wrote the first version of this paper, we feared that the problems presented were so difficult that young researchers reading the list might be discouraged rather than inspired. Happily, despite the difficulties, eight years has brought considerable progress on a number of these problems. Even for the two most central problems in the field, primality testing and factoring, there has been impressive progress: the primes are now known to be decidable in random polynomial time and the 'number field sieve' has given us the most powerful factoring algorithms yet. To emphasize the progress that has been made, the statement of each problems is followed by the original 1986 remarks and then the remarks which now seem appropriate.

The authors would appreciate your comments, particularly with regard to further progress on these problems.

Definitions, notation, and conventions.

In this paper:

- $\mathbb{R}$ denotes the set of real numbers,
- $\mathbb{Z}$ denotes the set of integers,
- $\mathbb{N}$ denotes the set of positive integers,
- *Primes* denotes the set of primes in $\mathbb{N}$,
- *Squarefrees* denotes the set of squarefree numbers in $\mathbb{N}$,
- $\mathbb{Q}$ denotes the set of rationals.
- ERH refers to the extended Riemann hypothesis.

For $a, b \in \mathbb{Z}$,

- we write $a \mid b$ if there exists $k \in \mathbb{Z}$ with $b = ka$,
- we write $a \not| b$ if there does not exist $k \in \mathbb{Z}$ with $b = ka$,
- $\gcd(a, b)$ denotes the greatest common divisor of a and b,
- $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol if b is odd and $\gcd(a, b) = 1$,
- $\langle a, b\rangle$ denotes the ordered pair.

For $n \in \mathbb{N}$,

- $\mathbb{Z}/n\mathbb{Z}$ denotes the ring of integers modulo n,
- $(\mathbb{Z}/n\mathbb{Z})^*$ denotes the corresponding multiplicative group,
- $\phi(n)$ denotes the number of elements in $(\mathbb{Z}/n\mathbb{Z})^*$,
- $L(n)$ represents any function of the form

$$\exp((1 + o(1))(\log n \log\log n)^{1/2}) \ .$$

- For $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, $L_n[\alpha, \beta]$ represents any function of the form

$$\exp((\beta + o(1))((\log n)^{\alpha}(\log\log n)^{1-\alpha})) \ .$$

If R is a ring, then we write $R[x]$ for the ring of polynomials with coefficients in R. The set of finite strings composed of the letters a and b is denoted $\{a, b\}^*$. For n, a, $b \in \mathbb{N}$ with $\gcd(n, 4a^3 + 27b^2) = 1$, let

$$S_{n,a,b} = \{\langle x, y\rangle \mid x, y \in \mathbb{Z}/n\mathbb{Z} \ \& \ y^2 \equiv x^3 + ax + b \pmod{n}\} \cup \{0\} \ .$$

When $p \in Primes$, $S_{p,a,b}$ is well known to be endowed with a group structure. We denote this group by $E_{p,a,b}$ and use $\#E_{p,a,b}$ for the number of elements of this group. More generally, if S is a set, we write $\#S$ for the cardinality of S.

In stating open problems we have decided to continue the ad hoc notation from [AM86]. For example, we label the first computational problem as **C1**, the corresponding open problem as **O1** (or **O1a** and **O1b** if there are two), and the original 1986 remarks concerning **C1** and **O1** we label as $\mathbf{Rem1}_{86}$. Any new remarks we label as $\mathbf{Rem1}_{94}$. Any additional references are given in **Ref1**. Computational problems **C2** and **C6** are stated in terms of a parameter S which is an arbitrary subset of $\mathbb{N}$. Computational problem **C30** is stated in terms of a parameter $c \in \mathbb{N}$.

While it seems inappropriate to spend a great deal of time giving rigorous definitions of the complexity-theoretic notions used in this paper, it seems worthwhile to provide some guidance. More information on these notions may be found in [Gil77], [AHU74], [AM77], and [GJ79]. We assume the concept of a polynomial time computable function is understood. A computational problem **C** is thought of as a set of pairs $\langle x, S_x\rangle$, where x is an input for which an output is desired and S_x is the set of possible 'correct' outputs on input x. For example

$$\begin{aligned}
\mathbf{C1} &= \{\langle n, S_n\rangle \mid n \in \mathit{Primes} \Rightarrow S_n = \{1\} \;\&\; n \notin \mathit{Primes} \Rightarrow S_n = \{0\}\} \\
\mathbf{C17} &= \{\langle\langle p, d\rangle, S_{\langle p,d\rangle}\rangle \mid d \in \mathbb{N} \;\&\; p \in \mathit{Primes} \;\&\; \\
&\qquad S_{\langle p,d\rangle} = \{f \mid f \in (\mathbb{Z}/p\mathbb{Z})[x] \mid \deg(f) = d \;\&\; f \text{ irreducible}\}\}. \\
\mathbf{C19} &= \{\langle p, S_p\rangle \mid p \in \mathit{Primes} \;\&\; S_p = \{g \mid g \in \mathbb{N}, 1 \le g \le p-1 \;\&\; \\
&\qquad g \text{ generates } (\mathbb{Z}/p\mathbb{Z})^*\}\} \\
\mathbf{C28} &= \{\langle\langle a, b, p, P, Q\rangle, S_{\langle a,b,p,P,Q\rangle}\rangle \mid a, b \in \mathbb{N}, p \in \mathit{Primes}, P, Q \in E_{p,a,b}, \\
&\qquad (\exists n \in \mathbb{N})[nP = Q] \;\&\; S_{\langle a,b,p,P,Q\rangle} = \{n \mid n \in \mathbb{N} \;\&\; nP = Q\}\}
\end{aligned}$$

Definition 1 *If* $\mathbf{C} = \{\langle x, S_x\rangle\}$ *is a computational problem then we let* $\pi(\mathbf{C}) = \{x \mid \langle x, S_x\rangle \in \mathbf{C}\}$.

We use $|x|$ to denote the length of an object x, where we hope that the meaning of 'length' will be clear from the context.

Definition 2 **C** *is in* $\mathcal{P}$ *iff there exists a polynomial time computable function* f *such that* $(\forall x \in \pi(\mathbf{C}))[f(x) \in S_x]$.

Thus for example, in **O18** below we ask if **C18** is in $\mathcal{P}$. Any deterministic algorithm which runs in polynomial time with input-output behaviour consistent with that described in **C18** would provide an affirmative answer to **O18**. In particular how that algorithm behaves on an input $p \notin \mathit{Primes}$ is irrelevant.

Definition 3 **C** *is in* $\mathcal{R}$ *iff there exists a* c *in* $\mathbb{N}$ *and a polynomial time computable function* f *such that*

i. $(\forall x \in \pi(\mathbf{C}))(\forall |r| \le |x|^c)[f(x, r) \in S_x \text{ or } f(x, r) = \text{"?"}]$

ii. $(\forall x \in \pi(\mathbf{C}))\left[\frac{\#\{r \mid |r| \le |x|^c \;\&\; f(x,r) \in S_x\}}{\#\{r \mid |r| \le |x|^c\}} \ge \frac{1}{2}\right]$

Definition 4 **C** *is in* $\mathcal{NP}$ *iff there exists a* c *in* $\mathbb{N}$ *and a polynomial time computable function* f *such that*

i. $(\forall x \in \pi(\mathbf{C}))(\forall |r| \leq |x|^c)[f(x,r) \in S_x \text{ or } f(x,r) = \text{"?"}]$.
ii. $(\forall x \in \pi(\mathbf{C}))(\exists y \in S_x)(\exists |r| \leq |x|^c)[f(x,r) = y]$.

Definition 5 **C** *is recognized in* $\mathcal{R}$ *iff*

i. $(\forall x \in \pi(\mathbf{C}))[S_x = \{1\} \Rightarrow (\forall |r| \leq |x|^c)[f(x,r) = \{1\} \text{ or } f(x,r) = \text{"?"}]]$
ii. $(\forall x \in \pi(\mathbf{C}))\left[S_x = \{1\} \Rightarrow \frac{\#\{r \mid |r| \leq |x|^c \ \&\ f(x,r)=1\}}{\#\{r \mid |r| \leq |x|^c\}} \geq \frac{1}{2}\right]$
iii. $(\forall x \in \pi(\mathbf{C}))[S_x \neq \{1\} \Rightarrow (\forall |r| \leq |x|^c)[f(x,r) = \text{"?"}]]$.

Definition 6 **C** *is recognized in* $\mathcal{NP}$ *iff there exists a* c *in* $\mathbb{N}$ *and a polynomial time computable function* f *such that*

i. $(\forall x \in \pi(\mathbf{C}))[S_x = \{1\} \Rightarrow (\forall |r| \leq |x|^c)[f(x,r) = \{1\} \text{ or } f(x,r) = \text{"?"}]]$
ii. $(\forall x \in \pi(\mathbf{C}))[S_x = \{1\} \Rightarrow (\exists |r| \leq |x|^c)[f(x,r) = 1]]$
iii. $(\forall x \in \pi(\mathbf{C}))[S_x \neq \{1\} \Rightarrow (\forall |r| \leq |x|^c)[f(x,r) = \text{"?"}]]$.

For notions involving the reduction of one problem to another we will be even less formal.

Definition 7 f *is a deterministic solution to* **C** *iff* $(\forall x \in \pi(\mathbf{C}))[f(x) \in S_x]$.

Let $D(\mathbf{C}) = \{f \mid f \text{ is a deterministic solution to } \mathbf{C}\}$. For all deterministic algorithms $\mathcal{A}$ and functions f and g, we say that $\mathcal{A}$ translates f into g iff when given a subroutine for f, $\mathcal{A}$ computes g in polynomial time (where the time used in the subroutine for f is not counted). We remark that calls to the subroutine may be 'dovetailed' but the algorithm A cannot know if the absence of a response on a particular call means that no response is forthcoming or that a response has just not arrived yet. See **C18** for an example.

Definition 8 $\mathbf{C1} \leq_{\mathcal{P}} \mathbf{C2}$ *iff there exists a deterministic algorithm* $\mathcal{A}$ *such that for all* $f \in D(\mathbf{C2})$, *there exists a* $g \in D(\mathbf{C1})$ *such that* $\mathcal{A}$ *translates* f *into* g *in polynomial time.*

Definition 9 **C** *is* $\mathcal{NP}$-hard *with respect to* $\mathcal{P}$ *iff for all* $\mathbf{C}'$, ($\mathbf{C}'$ is in $\mathcal{NP}$) $\Rightarrow$ ($\mathbf{C}' \ \leq_{\mathcal{P}} \mathbf{C}$).

We will follow the convention of using $\mathcal{NP}$-hard to denote $\mathcal{NP}$-hard with respect to $\mathcal{P}$.

Definition 10 f *is a random solution to* **C** *iff there exists a* c *in* $\mathbb{N}$ *such that*

i. $(\forall x \in \pi(\mathbf{C}))(\forall |r| \leq |x|^c)[f(x,r) \in S_x \text{ or } f(x,r) = \text{"?"}]$
ii. $(\forall x \in \pi(\mathbf{C}))\left[\frac{\#\{r \mid |r| \leq |x|^c \ \&\ f(x,r) \in S_x\}}{\#\{r \mid |r| \leq |x|^c\}} \geq \frac{1}{2}\right]$

Let $R(\mathbf{C}) = \{f \mid f \text{ is a random solution to } \mathbf{C}\}$.

Definition 11 $\mathbf{C1} \leq_{\mathcal{R}} \mathbf{C2}$ *iff there exists a deterministic algorithm* $\mathcal{A}$ *such that for all* $f \in D(\mathbf{C2})$, *there exists a* $g \in R(\mathbf{C1})$ *such that* $\mathcal{A}$ *translates* f *into* g *in polynomial time.*

Definition 12 **C** *is* $\mathcal{NP}$-hard *with respect to* $\mathcal{R}$ *iff for all* $\mathbf{C}'$,

$$(\mathbf{C}' \text{ is in } \mathcal{NP}) \Rightarrow \mathbf{C}' \ \leq_{\mathcal{R}} \mathbf{C} \ .$$

1 Primality testing

C1 Input $n \in \mathbb{N}$
Output 1 if $n \in \mathit{Primes}$,
0 otherwise.

O1a Is **C1** in $\mathcal{P}$?

O1b Is **C1** recognized in $\mathcal{R}$?

Rem1$_{86}$ A classical problem. The following quote appears in art. 329 of Gauss' *Disquisitiones Arithmeticæ*:(translation from [Knu81, page 398])

> The problem of distinguishing prime numbers from composites, and of resolving composite numbers into their prime factors, is one of the most important and useful in all of arithmetic. ...The dignity of science seems to demand that every aid to the solution of such an elegant and celebrated problem be zealously cultivated.

It is known that the set of composites is recognized in $\mathcal{R}$ [SS77]. If the extended Riemann hypothesis for Dirichlet L-functions is true, then **C1** is in $\mathcal{P}$ [Mil76]. There exists a constant $c \in \mathbb{N}$ and a deterministic algorithm for **C1** with running time $O((\log n)^{c \log\log\log n})$ [APR83]. If Cramér's conjecture on the gaps between consecutive primes is true, then **C1** is recognized in $\mathcal{R}$ [GK86]. **C1** is recognized in $\mathcal{NP}$ [Pra75]. Fürer [Für85] has shown that the problem of distinguishing between products of two primes that are $\not\equiv 1 \pmod{24}$ and primes that are $\not\equiv 1 \pmod{24}$ is in $\mathcal{R}$.

Rem1$_{94}$ Problem **O1b** has been settled in the affirmative by Adleman and Huang [AH92]. As a result of the work of H. Maier on gaps between consecutive primes, the exact formulation of Cramér's conjecture has now been called into question, however the conjecture required for [GK86] is unaffected.

Ref1 [Guy77], [Knu81], [Len81], [CL84], [Pom81], [Rab80a], [Rie85b], [Rie85a], [Wil78].

2 Testing an infinite set of primes

Let $S \subset \mathbb{N}$.

C2 Input $n \in \mathbb{N}$.
Output 1 if $n \in S$,
0 otherwise.

O2 Does there exist an infinite set $S \subset \mathit{Primes}$ such that **C2** is in $\mathcal{P}$?

Rem2$_{86}$ In light of **Rem1**$_{86}$ it is remarkable that **O2** remains unsettled. The related problem of the existence of an infinite set $S \subset \mathit{Primes}$ such that **C2** is recognized in $\mathcal{R}$ is addressed in [GK86].

Rem2$_{94}$ Problem **O2** been settled in the affirmative by Pintz, Steiger, and Szemerédi [PSS89]. One can now ask what the densest such set S is. In this direction, Konyagin and Pomerance [KP94] have proved that for every $\epsilon > 0$ there exists an algorithm that will prove primality in deterministic polynomial time for at least $x^{1-\epsilon}$ primes less than x.

Ref2 [PSS88].

3 Prime greater than a given bound

C3 Input $n \in \mathbb{N}$.
Output $p \in \mathit{Primes}$ with $p > n$.

O3 Is **C3** in $\mathcal{P}$?

Rem3$_{86}$ If Cramér's conjecture (see [Cra36]) on the gaps between consecutive primes is true, then **C3** $\leq_{\mathcal{P}}$**C1**. Since the density of primes between n and $2n$ is approximately $1/\log n$, it follows that **C3** $\leq_{\mathcal{R}}$**C1**. This problem has cryptographic significance [DH76], [RSA78].

Rem3$_{94}$ As we mentioned in **Rem1**$_{94}$, the exact formulation of Cramér's conjecture has now been called into question. It is still probably true that for every constant $c > 2$, there is a constant $d > 0$ such that there is a prime between x and $x + d(\log x)^c$. This hypothesis still implies that **C3** $\leq_{\mathcal{P}}$**C1**.

Note, since **C1** is recognized in $\mathcal{R}$ (see **Rem1**$_{94}$), it follows that **C3** is in $\mathcal{R}$. If anything, the importance of this problem has grown since 1986, since there have been numerous cryptosystems proposed since then that require the ability to construct large primes, sometimes with special properties. See [Pom90].

Ref3 [Bac88], [Pla79]. See also **Ref1**.

4 Prime in an arithmetic progression

C4 Input $a, n \in \mathbb{N}$.
Output $p \in \mathit{Primes}$ with $p \equiv a \pmod{n}$ if $\gcd(a, n) = 1$.

O4 Is **C4** in $\mathcal{P}$?

Rem4$_{86}$ It was conjectured by Heath-Brown [HB78] that if $\gcd(a, n) = 1$, then the least prime $p \equiv a \pmod{n}$ is $O(n \log^2 n)$, and this would imply that **C4** $\leq_{\mathcal{P}}$**C1**. If there are no Siegel zeroes, then the density of small primes in the arithmetic progression a modulo n is sufficient to conclude that **C4** $\leq_{\mathcal{R}}$**C1** [Bom74]. Without hypothesis, it is known [EH71] that Heath-Brown's conjecture is true for almost all pairs a, n with $\gcd(a, n) = 1$. Hence if **C1** is in $\mathcal{P}$, then one can solve **C4** in deterministic polynomial time for almost all inputs. See also **Rem20**$_{86}$.

Rem4$_{94}$ Since **C1** is now known to be in $\mathcal{R}$ (see **Rem1**$_{94}$), it follows that **C4** is also in $\mathcal{R}$. **C4** also has cryptographic applications [Sch91], [BM92], [oC91].

Ref4 [AM77]

5 Integer factoring

C5 Input $n \in \mathbb{N}$.
Output $p_1, p_2, \ldots, p_k \in \mathit{Primes}$ and $e_1, e_2, \ldots, e_k \in \mathbb{N}$ such that

$$n = \prod_{i=1}^{k} p_i^{e_i} \text{ if } n > 1 \ .$$

O5a Is **C5** in $\mathcal{P}$?

O5b Is **C5** in $\mathcal{R}$?

Rem5$_{86}$ Another classical problem, mentioned by Gauss in his *Disquisitiones Arithmeticæ* (see **Rem1**$_{86}$). There are a large number of random algorithms for **C5** whose running time is believed to be $L(n)^c$ for varying constants $c \geq 1$ [Pom82], [Len87], [SL84]. The only random algorithm of this class whose running time has actually been proved to be $L(n)^c$ is due to Dixon [Dix81]. Dixon's algorithm is unfortunately not practical. A determination of the complexity of **C5** would have significance in cryptography [RSA78].

Rem5$_{94}$ A great deal of progress has been made in the area of factoring integers. Lenstra and Pomerance [LP92] proved the existence of a probabilistic algorithm for factoring integers with an expected running time of $L_n[1/2, 1]$, improving on Dixon's bound. Another interesting development was the discovery of the number field sieve. A heuristic analysis suggests that there exists a constant $c > 0$ such that the number field sieve factors an integer n in expected time $L_n[1/3, c]$. Contributions to the number field sieve were made by a number of researchers, including (but not limited to) Adleman, Buhler, Coppersmith, Couveignes, A.K. Lenstra, H.W. Lenstra, Manasse, Odlyzko, Pollard, Pomerance and Schroeppel. See [Adl91], [Cop90], [Cou93], [LL93], and the references cited therein.

In a very recent development Peter Shor [Shoar] has shown that factoring can be done in polynomial time on a "quantum computer". It is premature to judge the implications of this development.

Ref5 [Dix81], [Guy77], [Knu81], [Len87], [MB75], [Pom82], [Rie85b], [Rie85a], [Sha71], [Sch82], [SL84], [Wil84].

6 Factoring a set of positive density

Let $S \subset \mathbb{N}$.

C6 Input $n \in \mathbb{N}$.

Output $p_1, p_2, \ldots, p_k \in \mathit{Primes}$ and $e_1, e_2, \ldots, e_k \in \mathbb{N}$ such that

$$n = \prod_{i=1}^{k} p_i^{e_i} \text{ if } n > 1 \text{ and } n \in S \ .$$

O6 Does there exist a set S such that

$$\liminf_{x\to\infty} \frac{\#\{n \mid n \le x \ \& \ n \in S\}}{x} > 0$$

and **C6**(S) is in $\mathcal{P}$?

Rem6$_{86}$ Assuming the necessary hypotheses for the running time analysis for Lenstra's elliptic curve factoring method (see [Len87]), it is probably possible to prove that a set S satisfying

$$\liminf_{x\to\infty} \frac{\#\{n \mid n \le x \ \& \ n \in S\}}{\frac{x \log\log^2 x}{\log x \log\log\log x}} > 0 \qquad (1)$$

can be factored in random polynomial time. This set will still have density zero, however. A related question is whether factoring a set of positive density is random polynomial time equivalent to **C5**. The set *Squarefrees* has density $6/\pi^2$ however it is not even clear that **C5** $\le_{\mathcal{R}}$ **C6**(*Squarefrees*).

Rem6$_{94}$ Let A denote a deterministic algorithm for factoring integers, and define $F(x, t, A)$ to be the number of integers n with $1 \le n \le x$ such that A will factor n in at most t bit operations. **O6** can then be stated as asking whether there exists an algorithm A and a constant $c > 0$ such that

$$\liminf_{x\to\infty} \frac{F(x, \log^c x, A)}{x} > 0 \ .$$

This problem remains open, but Hafner and McCurley [HM89a] and later Sorenson [Sor90] proved several results about the behaviour of F for various factoring algorithms (including a generalization to cover probabilistic algorithms). The estimate (1) has still not been proved, and the best result known [HM89a] in this direction is

$$F(x, \log^c x, A) >>_c \frac{x(\log\log x)^{\frac{6}{5}-\epsilon}}{\log x} \ ,$$

using a probabilistic algorithm. In this formulation, one may also ask for the slowest growing function $t(x)$ such that there exists an algorithm A with

$$\liminf_{x\to\infty} \frac{F(x, t(x), A)}{x} > 0 \ .$$

7 Squarefree part

C7 Input $n \in \mathbb{N}$.
Output $r, s \in \mathbb{N}$ with $n = r^2 s$ and $s \in \mathit{Squarefrees}$.

O7a Is **C7** in $\mathcal{P}$?

O7b Is **C5** $\leq_{\mathcal{R}}$ **C7** ?

Rem7$_{86}$ See **Rem13$_{86}$**. Clearly **C7** $\leq_{\mathcal{P}}$ **C5**. The analogous question for $f \in \mathbb{Q}[x]$ or $(\mathbb{Z}/p\mathbb{Z})[x]$ is solvable in polynomial time by performing calculations of the form $\gcd(f, f')$, where f' is the (formal) derivative of f. (see [Knu81, page 421]).

Rem7$_{94}$ Landau [Lan88] proved that **C7** $\leq_{\mathcal{P}}$ **C23**. According to [Len92], Chistov [Chi89] has shown that **C7** is polynomial time equivalent to determining the ring of integers in a number field.

8 Squarefreeness

C8 Input $n \in \mathbb{N}$.
Output 1 if $n \in \mathit{Squarefrees}$,
0 otherwise.

O8 Is **C8** in $\mathcal{P}$?

Rem8$_{86}$ A generalization of this is, given n and $k \in \mathbb{N}$, to determine if n is divisible by the kth power of a prime. Another generalization is to output $\mu = \mu(n)$, where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if there exists a } p \in \mathit{Primes} \text{ with } p^2 \mid n, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

Shallit and Shamir have shown that this generalization is reducible to the problem of computing the function d mentioned in **Rem9$_{86}$**.

Rem8$_{94}$ We are unaware of any progress on this problem.

9 Number of distinct prime factors

C9 Input $n \in \mathbb{N}$.
Output $\omega(n) = \#\{p \mid p \in \mathit{Primes} \ \& \ p \mid n\}$.

O9 Is **C9** in $\mathcal{P}$?

Rem9$_{86}$ Clearly **C1** $\leq_{\mathcal{P}}$ **C9**, since we can easily check to see if n is a perfect power. An interesting variant of **C9** is to output $\Omega(n) = e_1 + \ldots + e_k$, where $n = \prod_{i=1}^{k} p_i^{e_i}$ is the prime factorization of n. Another variant is to output $d(n) = \#\{k \mid k \in \mathbb{N} \ \& \ k \mid n\}$, and still another variant is to output the multiset $\{e_1, \ldots, e_k\}$. Shallit and Shamir [SS85] have proved that the last two variants are polynomial time equivalent to each other. As a consequence we have that **C9** is polynomial time reducible to the problem of computing the function $d(n)$ mentioned above.

Rem9$_{94}$ We are unaware of any progress on this problem. It is remarkable that one can decide if $\omega(n) = 1$ in random polynomial time [AH92], but there are no other partial results known on this problem.

10 Roots modulo a composite

C10 Input $e, a, n \in \mathbb{N}$.
Output $x \in \mathbb{N}$ such that $x^e \equiv a \pmod{n}$, if $\gcd(e, \phi(n)) = 1$ and $\gcd(a, n) = 1$.

O10 Is **C5** $\leq_{\mathcal{R}}$ **C10**?

Rem10$_{86}$ When the restriction that $\gcd(e, \phi(n)) = 1$ is dropped, it is known that **C5** $\leq_{\mathcal{R}}$ **C10** [Rab79]. A resolution of this problem would have important consequences in public-key cryptography [RSA78]. It is known that **C10** $\leq_{\mathcal{P}}$ **C23**.

Rem10$_{94}$ We are unaware of any progress on this problem.

11 Quadratic residuosity modulo a composite

C11 Input $a, n \in \mathbb{N}$.
Output 1 if there exists an $x \in \mathbb{N}$ such that $x^2 \equiv a \pmod{n}$ and $\gcd(a, n) = 1$,
0 otherwise.

O11a Is **C11** in $\mathcal{P}$?

O11b Is **C5** $\leq_{\mathcal{R}}$ **C11**?

Rem11$_{86}$ It is easy to show that **C11** $\leq_{\mathcal{P}}$ **C5**. There is an obvious generalization where the exponent 2 is replaced by another exponent k that is either fixed for the problem or supplied as an input. The presumed difficulty of **C11** has been used as a basis for cryptographic systems [GM82], [GM84], [Yao82], [BBS86]. **C11** is related to **C9** since the proportion of residues modulo n that are quadratic residues is $2^{-\omega(n)}$, where $\omega(n)$ is the number of distinct prime divisors of n. Therefore given an algorithm for **C11**, one can obtain a confidence interval for $\omega(n)$ by checking random values.

Rem11$_{94}$ We are unaware of any progress on this problem.

Ref11 [AM82].

12 Quadratic non-residue modulo a prime

C12 Input $p \in \mathbb{N}$.
Output $b \in \mathbb{N}$ such that there does not exist $c \in \mathbb{N}$ with $c^2 \equiv b \pmod{p}$, if $p \in \mathit{Primes}$.

O12 Is **C12** in $\mathcal{P}$?

Rem12$_{86}$ **C12** is easily seen to be in $\mathcal{R}$, since polynomial time algorithms for the corresponding problem of distinguishing quadratic residues from nonresidues can be based on the Jacobi symbol and the law of quadratic reciprocity, or else on Euler's criterion:

$$p \in \mathit{Primes} \text{ and } p \not| a \Rightarrow a^{\frac{p-1}{2}} \equiv (\frac{a}{p}) \pmod{p} \ .$$

Curiously, Gauss was aware of Euler's criterion, but was apparently unimpressed by its efficiency [Gau86, art. 106]:

> Although it is of almost no practical use, it is worthy of mention because of its simplicity and generality ... But as soon as the numbers we are examining are even moderately large this criterion is practically useless because of the amount of calculation involved.

Under the extended Riemann hypothesis, **C12** is in $\mathcal{P}$ [Mil76]. It is also known that the least quadratic nonresidue is almost always small [Erd61], so **C12** can be solved in deterministic polynomial time for almost all inputs.

Rem12$_{94}$ On the problem of calculating kth power non-residues in $\mathrm{GF}(p^n)$, the following is known. On ERH, the algorithm of Huang [Hua85], generalized by Evdokimov [Evd89], constructs a kth power non-residue, in $\mathrm{GF}(p^n)$ in deterministic time $(kn \log p)^{O(1)}$. Buchmann and Shoup [BS91], on ERH, construct a kth power non-residue in $\mathrm{GF}(p^n)$ in deterministic time $(\log p)^{O(n)}$. Bach [Bac90], on ERH, has given explicit bounds for estimations of the least kth power non-residue. See also **Rem19**$_{94}$.

Ref12 [Ank52], [Bac85].

13 Quadratic signature

C13 Input $\sigma \in \{-1, 1\}^*$.
Output The least $p \in \mathit{Primes}$ such that for all i with $1 \leq i \leq |\sigma|$, $(\frac{p_i}{p}) = \epsilon_i$, where $|\sigma|$, the length of σ, is the number of symbols in σ, p_i is the i^{th} prime, and ϵ_i is the i^{th} symbol of σ.

O13 Is **C13** in $\mathcal{P}$?

Rem13$_{86}$ If n has the form m^2q with q an odd prime and m odd, then for any a with $\gcd(a,n)=1$ we have $\left(\frac{a}{n}\right) = \left(\frac{a}{q}\right)$. It follows that if **C13** is in $\mathcal{P}$, then n could be partially factored since, assuming the extended Riemann hypothesis, q can be determined by a signature of length $O(\log^2 n)$ [Mil76], [Ank52]. The notion of quadratic signature can be generalized; see [AM82].

Rem13$_{94}$ The concept of quadratic signature has found application in the number field sieve [Adl91].

Ref13 [Ank52], [Bac85], [Bac90].

14 Square roots modulo a prime

C14 Input $a, p \in \mathbb{N}$.
Output $x \in \mathbb{N}$ with $x^2 \equiv a \pmod p$ if $p \in \mathit{Primes}$ and such an x exists.

O14 Is **C14** in $\mathcal{P}$?

Rem14$_{86}$ Among the researchers who have presented algorithms for **C14** are [Gau86, art. 319-322], [Ton91], [Leh69], [Sha72], [Ber67], [Rab80b], [AMM77]. It is now known that **C14** is in $\mathcal{R}$. It is also known that **C14** $\leq_{\mathcal{P}}$**C12** and that on the extended Riemann hypothesis, **C14** is in $\mathcal{P}$. There is a natural generalization of **C14** where the exponent 2 is replaced by a fixed k. Another generalization has k as part of the input. For this version there is a random time $O((k \log p)^c)$ algorithm based on known algorithms for **C15**. One can also use a discrete logarithm algorithm (see **Rem21$_{86}$**) to solve this variant, resulting in a random time $O(L(p))$ algorithm, which for large k will be faster.

Rem14$_{94}$ It is an oversight that we did not mention the work of Schoof [Sch85] on this problem in our earlier manuscript. Schoof proved that for fixed a, there exists a deterministic algorithm with running time polynomial in $\log p$.

Ref14 Many additional references are given in [LN83, page 182]. See also **Ref16** and [Hua85], [Evd89], [BS91].

15 Polynomial roots modulo a prime

C15 Input $p \in \mathbb{N}$, $f \in (\mathbb{Z}/p\mathbb{Z})[x]$.
Output $a \in \mathbb{Z}$ with $f(a) \equiv 0 \pmod p$ if $p \in \mathit{Primes}$ and such an a exists.

O15 Is **C15** in $\mathcal{P}$?

Rem15$_{86}$ See **Rem14$_{86}$**. **C15** is in $\mathcal{R}$ [Ber70], [CZ81], [Rab80b]. If the extended Riemann hypothesis is assumed and f has abelian Galois group over the rationals, then the problem is in $\mathcal{P}$ [Hua85].

Rem15$_{94}$ If f is fixed the problem appears to remain difficult; however, for certain f progress has been made. When f is linear the problem is trivial. When f is a quadratic there exists a deterministic polynomial time algorithm due to Schoof [Sch85]. When f is a cyclotomic polynomial, there exists a deterministic polynomial time algorithm due to Pila [Pil90].

Ref15 [Sho90b], [BS91]. See also **Ref16**.

16 Factoring polynomials modulo a prime

C16 Input $p \in \mathbb{N}, f \in (\mathbb{Z}/p\mathbb{Z})[x]$.
Output irreducible $g_1, \ldots, g_k \in (\mathbb{Z}/p\mathbb{Z})[x]$, and $e_1, \ldots, e_k \in \mathbb{N}$ such that $f = \prod_{i=1}^{k} g_i^{e_i}$, if $p \in \mathit{Primes}$.

O16 Is **C16** in $\mathcal{P}$?

Rem16$_{86}$ See **Rem15$_{86}$**. **C16** is in $\mathcal{R}$ [Ber70], [CZ81], [Rab80b]. The corresponding problem over $\mathbb{Q}$ is in $\mathcal{P}$ [LLL82].

Rem16$_{94}$ Let n denote the degree of f. Rónyai [Rón88] on ERH gives a deterministic algorithm with running time $(n^n \log p)^{O(1)}$. Evdokimov [Evdar] on ERH gives a deterministic algorithm with running time $(n^{\log n} \log p)^{O(1)}$. In particular, both algorithms are polynomial time if the degree is bounded. For the case $f \in Z[x]$, f irreducible and $Q[x]/(f)$ Abelian over Q, Huang [Hua91] on ERH gives a deterministic polynomial time algorithm. For the case $f \in Z[x]$, f irreducible and $Q[x]/(f)$ Galois over Q, Rónyai on ERH gives a deterministic polynomial time algorithm [Rón89]. For the case $f \in Z[x]$ solvable, Evdokimov [Evd89] on ERH gives a deterministic polynomial time algorithm.

Lenstra [Len90] has shown in many cases the assumption of ERH above may be removed if irreducible polynomials of appropriate degree can be found in deterministic polynomial time.

Buchmann and Shoup [BS91] proved, under ERH, that for all $n \in \mathbb{N}$, there exists a deterministic algorithm for **C16** with running time $\sqrt{k}$ times a polynomial in the input size, where k is the largest prime dividing $\phi_n(p)$ and ϕ_n is the n-th cyclotomic polynomial.

Ref16 [Ber67], [Ber68], [Knu81, pages 420–441], [LN83, pages 147-185].

17 Irreducible polynomials

C17 Input $d, p \in \mathbb{N}$.

Output irreducible $f \in (\mathbb{Z}/p\mathbb{Z})[x]$ with $degree(f) = d$, if $p \in Primes$.

O17 Is **C17** in $\mathcal{P}$?

Rem17$_{86}$ **C17** is in $\mathcal{R}$ [Ber68], [Rab80b]. **C17** is in $\mathcal{P}$ if the extended Riemann hypothesis is true [AL86]. There is a $c \in \mathbb{N}$ and a deterministic polynomial time algorithm which on input d, p with $p \in Primes$ outputs an irreducible $f \in (\mathbb{Z}/p\mathbb{Z})[x]$ of degree greater than $cd/\log p$ and less than or equal to d [AL86]. Since irreducible quadratics yield quadratic nonresidues, it is clear that **C12** $\leq_{\mathcal{P}}$**C17**, and also from the results on **C14** that **C14** $\leq_{\mathcal{P}}$**C17**.

Rem17$_{94}$ The result of [AL86] was discovered independently by Evdokimov [Evd89]. Shoup [Sho90a] proved **C17** $\leq_{\mathcal{P}}$**C16**, and gave a deterministic algorithm for finding an irreducible polynomial of degree d over $\mathbb{Z}/p\mathbb{Z}$ in time $\sqrt{p}(d + \log p)^{O(1)}$.

Ref17 [Len92].

18 Recognition of a primitive root modulo a prime

C18 Input $b, p \in \mathbb{N}$.
Output 1 if b is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$ and $p \in Primes$,
0 if b is not a generator of $(\mathbb{Z}/p\mathbb{Z})^*$ and $p \in Primes$.

O18a Is **C18** in $\mathcal{P}$?

O18b Is **C18** recognized in $\mathcal{R}$?

Rem18$_{86}$ It is known that **C18** $\leq_{\mathcal{P}}$**C5**, since b is a primitive root modulo p if and only if $p \not| b$ and

$$\forall q[[q \in Primes \ \&\ q \mid p-1] \Rightarrow b^{(p-1)/q} \not\equiv 1 \pmod{p}] \ .$$

A generalization of **C18** where a third input $c \in \mathbb{N}$ is given and the output is 1 if and only if b has order c is also of interest.

Rem18$_{94}$ We are unaware of any progress on this problem. We would like to point out however that under ERH, **C18** $\leq_{\mathcal{P}}$**C21**. To see why, recall that under ERH, the least primitive root modulo p is $\leq c\log^6 p$ for some constant c [Sho90c]. Let g be a suspected primitive root modulo p. We dovetail the following procedures:

process A for $b = 1, 2, \ldots, c\log^6 p$: ask oracle for **C21** to compute an x with $g^x \equiv b \pmod{p}$. If the oracle returns an x keep it only if you confirm that $g^x \equiv b \pmod{p}$. If for all b an x is kept then output "primitive root".

process B for $b = 1, 2, \ldots, c\log^6 p$: ask oracle for **C21** to compute x such that $b^x \equiv g \pmod{p}$. If the oracle returns an x keep it only if you confirm that $b^x \equiv g \pmod{p}$. If for some b an x is kept with $\gcd(x, p-1) > 1$, then output "not a primitive root".

19 Finding a primitive root modulo a prime

C19 Input $p \in \mathbb{N}$.
Output $g \in \mathbb{N}$ such that $1 \leq g \leq p-1$ and g generates $(\mathbb{Z}/p\mathbb{Z})^*$, if $p \in \mathit{Primes}$.

O19 Is **C19** in $\mathcal{P}$?

Rem19$_{86}$ The density of generators is sufficient that it is easily shown that **C19** $\leq_{\mathcal{R}}$ **C18**. If the extended Riemann hypothesis is true, then the least generator is small [Wan61], and **C19** $\leq_{\mathcal{P}}$ **C18**. An interesting variant of **C19** involves finding elements of $(\mathbb{Z}/p\mathbb{Z})^*$ of desired order. **C19** has an obvious extension to an arbitrary finite field, or for that matter to any cyclic group.

Rem19$_{94}$ Shoup [Sho90c] proved several results related to this problem. Among other things, he proved under the assumption of the extended Riemann hypothesis that a primitive root for $\mathrm{GF}(p^2)$ can be constructed in deterministic polynomial time. Buchmann and Shoup [BS91], on ERH, give a deterministic algorithm, which on input an irreducible f of degree n over $\mathbb{Z}/p\mathbb{Z}$, outputs a generating set for $\mathbb{Z}/p\mathbb{Z}[x]/(f)$ in time $(\log p)^{O(n)}$. As a consequence, if the factorization of $p^n - 1$ is known, then under the assumption of ERH, a primitive root of $\mathrm{GF}(p^n)$ can be computed in deterministic polynomial time.

20 Calculation of orders modulo a prime

C20 Input $a, p \in \mathbb{N}$.
Output $k = min\{x \mid x \in \mathbb{N}, a^x \equiv 1 \pmod{p}\}$, if $p \in \mathit{Primes}$ and $\gcd(a,p) = 1$.

O20 Is **C20** in $\mathcal{P}$?

Rem20$_{86}$ The variant in which p is not required to be prime is random polynomial time equivalent to **C5** [Mil76]. A related question: is the problem of factoring numbers of the form $p-1$, with p prime, polynomial time reducible to **C20**? If **C6** is in $\mathcal{P}$, then the problem of factoring numbers of the form $p-1$ with p prime is polynomial time equivalent to factoring.

Rem20$_{94}$ We are unaware of any progress on this problem.

21 Discrete logarithm modulo a prime

C21 Input $g, b, p \in \mathbb{N}$.
Output $x \in \mathbb{N}$ with $g^x \equiv b \pmod{p}$, if $p \in \mathit{Primes}$ and such an x exists.

O21 Is **C21** in $\mathcal{P}$?

Rem21$_{86}$ If the prime factors of $p-1$ are less than $\log^c p$ for some constant $c > 0$, then the problem is in $\mathcal{P}$ [PH78]. The fastest known algorithms for solving **C21** have running times of $L(p)$ [COS86]. The resolution of **O21** would have important consequences in cryptography [ElG85], [BM84]. There is an obvious generalization of **C21** to an arbitrary finite field. Bach [Bac84] has asked if the problem of factoring numbers of the form $p-1$, with p prime, is polynomial time reducible to **C21**.

Rem21$_{94}$ There has been considerable progress on this problem. Pomerance [Pom86] proved that there exists a probabilistic algorithm to compute discrete logarithms in GF(q) with expected running time of $L_q[1/2, \sqrt{2}]$, for the case where q is prime or q is a power of 2. Gordon [Gor93] presented an adaptation of the number field sieve to computing discrete logarithms in $\mathbb{Z}/p\mathbb{Z}$, along with a heuristic argument to suggest an expected running time of $L_p[1/3, c]$ for some positive constant c.

For discrete logarithms over general finite fields, progress has also been made. At the time that we wrote our original paper, we neglected to mention the work of Coppersmith [Cop84], who had published an algorithm for GF(2^n) with a heuristic expected running time bounded by $L_{2^n}[1/3, c]$ for some positive constant c. Lovorn [Lov92] proved a running time of $L_q[1/2, c]$ for some positive constant c when $q = p^n$ with $\log p \leq n^{0.98}$. Adleman and DeMarrais [AD93a] gave an algorithm for arbitrary finite fields whose heuristic expected running time is $L_q[1/2, c]$ for some positive constant c. Adleman's function field sieve [Adlar] gives a heuristic expected running time of $L_q[1/3, c]$ for some positive constant c when $q = p^n$ and $\log p \leq n^{g(n)}$, where g is any function such that $0 < g(n) < 0.98$ and $\lim_{n\to\infty} g(n) = 0$.

Surveys on the discrete logarithm problem have been published: [vO91], [McC90a], [Odl94].

Historically, advances in integer factoring algorithms have brought corresponding advances in discrete logarithm algorithms. The first author thinks it is an interesting research problem to establish whether reductions exist between **C5** and **C21**. The second author finds the evidence for the existence of such reductions to be unconvincing.

In a very recent development Peter Shor [Shoar] has shown that discrete logarithms can be computed in polynomial time on a "quantum computer". It is premature to judge the implications of this development.

Ref21 [Odl85], [Sch93], [AD93b].

22 Discrete logarithm modulo a composite

C22 Input $g, b, n \in \mathbb{N}$.
Output $x \in \mathbb{N}$ with $g^x \equiv b \pmod{n}$, if such an x exists.

O22a Is **C22** in $\mathcal{P}$?

O22b Is **C5** $\leq_{\mathcal{P}}$**C22**?

Rem22$_{86}$ Clearly **C21** $\leq_{\mathcal{P}}$**C22**. It is also known that **C5** $\leq_{\mathcal{R}}$**C22** [Bac84]. The resolution of **O22** would have consequences in public-key cryptography [McC88]. There is an obvious generalization to an arbitrary group (see also **C28**).

Rem22$_{94}$ We are unaware of any progress on this problem.

23 Calculation of $\phi(n)$

C23 Input $n \in \mathbb{N}$.
Output $\phi(n)$.

O23 Is **C5** $\leq_{\mathcal{P}}$**C23**?

Rem23$_{86}$ It is known that **C5** $\leq_{\mathcal{R}}$**C23** [Mil76], and it is obvious that **C23** $\leq_{\mathcal{P}}$**C5**. **C5** is known to be random polynomial time equivalent to the problem of computing $\sigma(n)$, the sum of the positive integral divisors of n [BMS84].

Rem23$_{94}$ We are unaware of any progress on this problem. See **Rem7**$_{94}$.

24 Point on an elliptic curve

C24 Input $a, b, p \in \mathbb{N}$.
Output $x, y \in \mathbb{N}$ with $y^2 \equiv x^3 + ax + b \pmod{p}$, if $p \in \mathit{Primes}$ and $p \nmid 4a^3 + 27b^2$.

O24 Is **C24** in $\mathcal{P}$?

Rem24$_{86}$ One can show that **C24** is in $\mathcal{R}$, since there is an easy argument to show that **C24** $\leq_{\mathcal{R}}$**C14**: choose random values of x, evaluate the right hand side, and use a random algorithm for **C14** to try to solve for y. A theorem of Hasse implies that the probability of choosing a successful x is approximately $\frac{1}{2}$.

Rem24$_{94}$ We are unaware of any progress on this problem. **C24** has applications in cryptography [Kob87b, p. 162].

25 Binary quadratic congruences

C25 Input $k, m, n \in \mathbb{N}$.

Output $x, y \in \mathbb{N}$ with $x^2 - ky^2 \equiv m \pmod{n}$, if n is odd and $\gcd(km, n) = 1$.

O25 Is **C25** in $\mathcal{P}$?

Rem25$_{86}$ **C25** is in $\mathcal{R}$ [AEM87]. If the extended Riemann hypothesis and Heath-Brown's conjecture on the least prime in an arithmetic progression are true, then **C25** is in $\mathcal{P}$ [Sha84]. **C25** arose from cryptography [OSS84], [PS87]. In fact, **C25** is only one example of a wide range of questions concerning solutions of $f \equiv 0 \pmod{n}$, where f is a multivariate polynomial with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Such questions can vary greatly in their complexity as the form of the question changes. We may ask questions about determining if a solution exists, finding a solution, finding the least solution, or finding the number of solutions. We may vary the form of the polynomial or the properties of n (e.g. prime, composite, squarefree). As an example of the variation in complexity, even for the polynomial $f(x) = x^2 - a$ we have the following situation:

1. The problem of deciding from inputs $a, p \in \mathbb{N}$ whether $x^2 - a \equiv 0 \pmod{p}$ has a solution when p is prime is in $\mathcal{P}$ (see **Rem12**$_{86}$.)
2. The problem of finding from inputs $a, p \in \mathbb{N}$ a solution of $x^2 - a \equiv 0 \pmod{p}$ when p is prime is in $\mathcal{R}$ (see **Rem14**$_{86}$).
3. The problem of finding from inputs $a, n \in \mathbb{N}$ a solution of $x^2 - a \equiv 0 \pmod{n}$ is random equivalent to the problem of factoring n (see **Rem10**$_{86}$).
4. The problem of finding from inputs $a, n \in \mathbb{N}$ the least positive integer solution of $x^2 - a \equiv 0 \pmod{n}$ is $\mathcal{NP}$-hard [MA78].

We therefore view the problem of classifying all problems concerning solutions of $f \equiv 0 \pmod{n}$ according to their complexity as an important metaproblem.

Rem25$_{94}$ We are unaware of any progress on this problem. There has been marginal progress on the "metaproblem". We regard this area as a very fruitful one for future investigations.

Ref25 [vzGKS93]. Some cryptographic problems related to the metaproblem are mentioned in [McC90b]. That paper also contains pointers to other unsolved number-theoretic problems relating to cryptology.

26 Key distribution

C26 Input $g, p, a, b \in \mathbb{N}$.
Output $c \in \mathbb{N}$, where $c \equiv g^{xy} \pmod{p}$, if $p \in \mathit{Primes}$, g is a primitive root modulo p, $a \equiv g^x \pmod{p}$, and $b \equiv g^y \pmod{p}$.

O26 Is **C21** $\leq_{\mathcal{R}}$**C26**?

Rem26$_{86}$ The motivation for this problem comes from cryptography [DH76]. It is obvious that **C26** $\leq_{\mathcal{P}}$ **C21**. There is a generalization where p is replaced by a composite n, and we ask only for an output c when a and b are powers of g. For this generalization is the problem equivalent to **C5** or **C22** (see [Bac84], [McC88])?

Rem26$_{94}$ Bert den Boer [dB90] proved that when all prime factors of $\phi(p-1)$ are small, the key distribution problem is as hard as computing discrete logarithms.

Ref26 [Odl85], [ElG85].

27 Construction of an elliptic curve group of a given order

C27 Input $p, n \in \mathbb{N}$.
Output $a, b \in \mathbb{N}$ with $\#E_{p,a,b} = n$, if $p \in \mathit{Primes}$ and such an a, b exist.

O27 Is C27 in $\mathcal{P}$?

Rem27$_{86}$ There is a polynomial time algorithm that, given p, a, and b with $p \not| 4a^3 + 27b^2$ computes $\#E_{p,a,b}$ [Sch85].

Rem27$_{94}$ We are unaware of any progress on this problem, however it is known that for some primes p, supersingular curves of order $p+1$ can be constructed efficiently (see [MOV94]).

Ref27 [Kob87b], [Kob87a], [Sch85], [Sil86], [Kob91], [Kob91], [Kob88].

28 Discrete logarithms in elliptic curve groups

C28 Input $a, b, p \in \mathbb{N}, P, Q \in S_{p,a,b}$
Output $n \in \mathbb{N}$ with $P = nQ$, if $p \in \mathit{Primes}$ and such an n exists.

O28 Is **C28** in $\mathcal{P}$?

Rem28$_{86}$ The presumed difficulty of this problem has been used as the basis for a public key cryptosystem and digital signature scheme [Kob87b], [Mil86]. Whereas for the discrete logarithm problem in the multiplicative group modulo a prime there is a subexponential algorithm (see **Rem21**$_{86}$), no such algorithm is known to exist for **C28**. A related problem is given a, b, and p to construct a minimal set of generators for $E_{p,a,b}$.

Rem28$_{94}$ Menezes, Okamoto, and Vanstone [MOV94] used Weil pairing to prove that there exists a probabilistic reduction from **C28** to the problem of computing discrete logarithm in the multiplicative group of a (perhaps high degree) extension of GF(q). For supersingular curves, this reduction can be carried out in random polynomial time, with the result that a probabilistic subexponential algorithm is obtained for **C28** in this special case.

Koblitz [Kob90] has suggested cryptographic uses for the rational subgroups of the Jacobian of a hyperelliptic curve over a finite field. Adleman, Huang, and DeMarrais [AHDar] discovered a heuristic subexponential probabilistic algorithm for the discrete logarithm problem in these subgroups when the genus of the curve is large with respect to the size of the finite field.

29 Shortest vector in a lattice

C29 Input $b_1, \ldots, b_n \in \mathbb{Z}^n$
Output $v \in \Lambda$ with $||v||_2 = \min\{||x||_2 \mid x \in \Lambda, x \neq 0\}$, where $\Lambda = \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_n$ if $b_1, \ldots, b_n$ span $\mathbf{R}^n$.

O29 Is **C29** $\mathcal{NP}$-hard?

Rem29$_{86}$ The corresponding problems with norms $||\cdot||_\infty$ and $||\cdot||_1$ are known to be $\mathcal{NP}$-hard [Lag85], [vEB81]. See also **Rem30**$_{86}$.

Rem29$_{94}$ It was an oversight that we did not mention the result of Lenstra [Len83], who proved that if the dimension n is fixed, the shortest vector in a lattice of dimension n can be found in polynomial time.

Ref29 [GLS88] and [Lov86] contain nice surveys of this and related topics.

30 Short vector in a lattice

Let $c \in \mathbb{N}$

C30 Input $b_1, \ldots, b_n \in \mathbb{Z}^n$
Output $v \in \Lambda$ with $||v||_2 \leq n^c \min\{||x||_2 \mid x \in \Lambda, x \neq 0\}$, where $\Lambda = \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_n$ if $b_1, \ldots, b_n$ span $\mathbf{R}^n$.

O30 Does there exist a $c \in \mathbb{N}$ for which **C30** is in $\mathcal{P}$?

Rem30$_{86}$ In [LLL82] it was shown that there is a polynomial time algorithm that produces a vector $v \in \Lambda$ with

$$||v||_2 \leq 2^{\frac{n-1}{2}} \min\{||x||_2 \mid x \in \Lambda, x \neq 0\} \ ,$$

and in [Sey87] it was shown that for any $\epsilon > 0$ there is a polynomial time algorithm $\mathcal{A}_\epsilon$ that produces a vector $v \in \Lambda$ with

$$||v||_2 \leq (1+\epsilon)^n \min\{||x||_2 \mid x \in \Lambda, x \neq 0\} \ .$$

A number of related problems in simultaneous diophantine approximation are discussed in [Lag85] and [Fru85].

Rem30$_{94}$ We are unaware of any progress on this problem.

Ref30 [LLS90], [GLS88], [Lov86].

31 Galois group of a polynomial

C31 Input $f \in \mathbb{Q}[x]$.
Output $n = [K : \mathbb{Q}]$, where K is the splitting field of f.

O31 Is **C31** in $\mathcal{P}$?

Rem31$_{86}$ n is the order of the Galois group associated with f. Polynomial time algorithms exist for determining if n is a power of 2 or if the Galois group is solvable [LM85]. Many other properties of the Galois group can also be determined in polynomial time [Kan85].

Rem31$_{94}$ Landau [Lan85] proved that the Galois group can be computed in deterministic time $O((\#G + \ell)^c)$ for some constant $c > 0$, where ℓ is the length of the input specification of f and K. Further results are discussed in [Len92], but the problem remains open.

32 Class numbers

C32 Input $d \in \mathbb{N}$.
Output $h(-d)$, the order of the group of equivalence classes of binary quadratic forms with discriminant $-d$ under composition.

O32 Is **C32** in $\mathcal{P}$?

Rem32$_{86}$ This is related to classical questions of Gauss [Gau86, art. 303]. It appears that the results of Shanks [Sha72], [Sha71], Schnorr & Lenstra [SL84], Seysen [Sey87], and Schoof [Sch82] establish that **C5** $\leq_{\mathcal{R}}$**C32**, and that ERH implies **C5** $\leq_{\mathcal{P}}$**C32**. It is remarked in [BMS84] that it is not even known if **C32** is in $\mathcal{NP}$. The best known algorithm for computing $h(-d)$ is due to Shanks [Sha71]. The question could also be stated in terms of the class number of orders in the field $\mathbb{Q}(\sqrt{-d})$.

Rem32$_{94}$ McCurley [McC89] proved under ERH that **C32** is in $\mathcal{NP}$. Hafner and McCurley [HM89b] proved under ERH that there exists a probabilistic algorithm with expected running time $L_d[1/2, \sqrt{2}]$ that will compute not only the class number $h(-d)$, but also the structure of the class group. These results were extended to the case of real quadratic fields by Buchmann and Williams [BW89]. Thiel [Thiar] has shown under ERH that verifying the class number belongs to $\mathcal{NP} \cap \mathrm{co}\mathcal{NP}$.

The more general question of computing class numbers and class groups of arbitrary algebraic number fields is also of interest. According to Lenstra [Len92], Buchmann and Lenstra proved that there is a deterministic exponential time algorithm for computing the cardinality and structure of the class group. Buchmann [Buc90] gave a probabilistic subexponential algorithm for a special case of this problem. Lenstra [Len92] outlines an approach to obtaining a probabilistic subexponential algorithm in the general case.

Lenstra's paper [Len92] is an important source for information concerning algorithms and open problems concerning algebraic number fields.

Ref32 [Gol85], [Sha72], [Sch82], [Lag80b], [Buc90].

33 Solvability of binary quadratic diophantine equations

C33 Input $a, b, c, d, e, f \in \mathbb{Z}$.
Output 1 if there exists $x, y \in \mathbb{Z}$ with $ax^2 + bxy + cy^2 + dx + ey + f = 0$ and there does not exist a $g \in \mathbb{Z}$ with $b^2 - 4ac = g^2$,
0 otherwise.

O33a Is **C33** $\mathcal{NP}$-hard?

O33b Is **C33** $\mathcal{NP}$-hard with respect to $\mathcal{R}$?

Rem33$_{86}$ It is known that **C33** is recognized in $\mathcal{NP}$ [Lag79]. Without the constraint that $b^2 - 4ac$ is not a square, the problem is known to be $\mathcal{NP}$-hard [MA78]. Certain variants of **C33** are known to be $\mathcal{NP}$-hard with respect to $\mathcal{R}$ [AM77].

Rem33$_{94}$ We are unaware of any progress on this problem.

34 Solvability of anti-Pellian equation

C34 Input $d \in \mathbb{N}$.
Output 1 if there exist $x, y \in \mathbb{Z}$ with $x^2 - dy^2 = -1$,
0 otherwise.

O34 Is **C34** in $\mathcal{P}$?

Rem34$_{86}$ There exist choices of d for which the smallest solution of $x^2 - dy^2 = -1$ cannot be written down in polynomial space [Lag79]. It is known that **C34** is in $\mathcal{NP}$ [Lag80a]. If the factorization of d is provided as part of the input, then the problem is recognized in $\mathcal{R}$, and if in addition we assume the extended Riemann hypothesis, then the problem is in $\mathcal{P}$ [Lag80a].

Rem34$_{94}$ We are unaware of any progress on this problem.

35 Greatest common divisors in parallel

C35 Input $a, b \in \mathbb{N}$.
Output $\gcd(a, b)$.

O35 Is **C35** in $\mathcal{NC}$?

Rem35$_{86}$ The best known results for computing greatest common divisors in parallel are contained in [BK83], [CG] and [KMR87]. One may ask a similar question for the modular exponentiation problem: given $a, b, n \in \mathbb{N}$, compute $a^b \pmod{n}$. For a definition of $\mathcal{NC}$ see [Coo85] or [Coo81].

Rem35$_{94}$ Polylog depth, subexponential size circuits for both integer GCD and modular exponentiation have been obtained by Adleman and Kompella [AK88].

Ref35 [KMR84].

36 Integer multiplication in linear time

C36 Input $a, b \in \mathbb{N}$.
Output ab.

O36 Does there exist an algorithm to solve **C36** that uses only $O(\log(ab))$ bit operations ?

Rem36$_{86}$ The best known algorithm is due to Schönhage and Strassen and uses $O(\log(ab) \cdot \log\log(ab) \cdot \log\log\log(ab))$ bit operations [SS71].

Rem36$_{94}$ We are unaware of any progress on this problem.

Ref36 [Knu81, pages 278-301]

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