The Distribution of r-Free Integers in Arithmetic Progressions

Kevin S. McCurley

1. INTRODUCTION.

A natural number is called r-free if it is not divisible by the rth power of a prime. Let $S_r(x;q,a)$ denote the number of r-free numbers in the arithmetic progression a modulo q that do not exceed x, and let

$$R_r(x;q,a) = S_r(x;q,a) - \frac{x}{q} f(a,q) ,$$

where

$$f(a,q) = \sum_{\substack{d=1 \\ (d^r,q)|a}}^{\infty} \frac{\mu(d)(d^r,q)}{d^r} .$$

We shall always assume that (a,q) is r-free, for otherwise $S_r(x;q,a)$ is zero.

In this paper, we shall be concerned with the estimation of $R_r(x;q,a)$, with emphasis on uniformity in a, q, and r. We shall use $c_1, c_2 \ldots$ to denote constants, and unless otherwise indicated all constants will be independent of a, q, and r. Our starting point is the formula

$$(1) \qquad S_r(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \sum_{d^r|n} \mu(d)$$

from which an elementary argument yields the estimate

$$(2) \qquad R_r(x;q,a) \ll x^{1/r} .$$

In the case $(a,q) = 1$ a more elaborate argument due to Prachar [6] yields

$$R_r(x;q,a) \ll r^{\omega(q)}\{x^{1/r}q^{-1/r^2} + q^{1/r}\} \quad , \tag{3}$$

where $\omega(q)$ is the number of distinct prime factors of q. For $r = 2$ this has been improved by Hooley [4] to

$$R_2(x;q,a) \ll x^{1/2}q^{-1/2} + q^{1/2+\epsilon} \quad , \tag{4}$$

and the recent work of Heath-Brown [3] would seem to imply a stronger result than (4), at least in the case $x \leq q^2$.

The previous results are primarily concerned with small values of x relative to q, whereas in this paper we shall be concerned with large values of x. Siebert [7] proved that if $\epsilon > 0$ is arbitrary and $x \geq \exp(q^\epsilon)$, then

$$R_r(x;q,a) \ll x^{1/r} \exp(-c_1(\epsilon,r)\sqrt{\log x}) \quad . \tag{5}$$

This result is analogous to the Siegel-Walfisz theorem for primes in arithmetic progressions (see Davenport [2], p. 132). The distribution of primes in arithmetic progressions modulo q depends on the location of zeros of Dirichlet L-functions formed with characters modulo q. We say that q is an exceptional modulus if there exists a real character modulo q such that the associated L-function has a real zero exceeding $1 - c_2/\log q$. Page [5] proved that if q is not an exceptional modulus, then the Siegel-Walfisz theorem can be substantially improved.

Our first result is an improvement of (5) that is analogous to Page's theorem.

THEOREM 1. There exist absolute computable constants c_3 and c_4 such that if $x \geq \exp(c_3 r \log^2 q)$ and q is not exceptional, then

$$R_r(x;q,a) \ll (xq)^{1/r} \exp(-c_4 r^{-3/2}\sqrt{\log x}) .$$

Note that this is inferior to (1) unless $x \geqslant \exp(c_4^{-2} r \log^2 q)$. Theorem 1 may also be regarded as a generalization of a result of Walfisz [8, pp. 192-198], who proved it for $q = 1$.

The proof of Theorem 1 is similar to that of Siebert [7], and is based on an estimate for the functions

$$M(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \mu(n)$$

This requires information concerning the zeros of all L-functions formed with characters modulo q. In our next theorem we use a slightly different method to show that we need only be concerned with characters of the form χ^r. In contrast with previous methods, we now assume that $(a,q) = 1$

THEOREM 2. Let q and r be such that $L(s,\chi^r)$ has no real zeros exceeding $1 - c_2/\log q$, for all χ modulo q. Then there exist constants c_6 and c_7 such that

$$R_r(x;q,a) \ll x^{1/r} \exp(-c_6 r^{-3/2}\sqrt{\log x}) ,$$

provided $x \geqslant \exp(c_7 r \log^2 q)$ and $(a,q) = 1$.

From Theorem 2 it is apparent that the problem of exceptional moduli for r-free numbers is different from that of primes. For example, if $r = \varphi(q)$ then χ^r is principal and $L(s,\chi^r)$ has no positive real zeros. For a given r, the moduli that are potentially troublesome for r-free numbers are those for which there exist characters χ with χ^r a quadratic character. In the case of squarefree numbers, this has the following consequence.

Corollary. There exist constants c_8 and c_9 such that if $x \geq \exp(c_8 \log^2 q)$, $(a,q) = 1$, $16 \nmid q$, and q is not divisible by a prime congruent to 1 modulo 4, then

$$R_2(x;q,a) \ll x^{1/2} \exp(-c_9\sqrt{\log x}) .$$

In order to prove the Corollary, we write $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and write a character χ modulo q as $\chi = \chi_1\chi_2\ldots\chi_k$, where χ_i is a character modulo $p_i^{\alpha_i}$. The order of χ is then the least common multiple of the order of the characters χ_i, so it suffices to prove that χ_i is not of order 4. If p_i is odd, then the order of χ_i divides $\varphi(p_i^{\alpha_i}) = (p_i - 1)p_i^{\alpha_i-1}$, and this is not divisible by 4 if $p_i \equiv 3 \pmod 4$. Finally we observe that there are no quartic characters modulo 1, 2, 4, or 8.

2. PRELIMINARIES.

The following result is analogous to a result of Page

Lemma 1. There exist constants c_{10} and c_{11} such that if $L(s,\chi)$ has no real zeros exceeding $1 - c_2/\log q$, and $x \geq \exp(c_{10}\log^2 q)$, then

$$\sum_{n\leq x} \mu(n)\chi(n) \ll x \exp(-c_{11}\sqrt{\log x}$$

The proof of Lemma 1 is omitted, since it is implicit in the work of Davenport [1]. From Lemma 1 we immediately obtain the following lemmas.

Lemma 2. If $x \geq \exp(c_{10}\log^2 q)$ and q is not exceptional, then

$$M(x;q,\ell) \ll x \exp(-c_{11}\sqrt{\log x})$$

Proof. If $(\ell,q) = 1$, then

$$M(x;q,\ell) = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(\ell) \sum_{n \leq x} \chi(n)\mu(n) ,$$

and the result follow from Lemma 1. The case $(\ell,q) > 1$ is similarly handled in Davenport [1].

Lemma 3. If $x \geq \exp(c_{10}\log^2 q)$, $r \geq 2$, and q is not exceptional, then

$$\sum_{\substack{n > x \\ n \equiv \ell \pmod q}} \mu(n)n^{-r} \ll x^{1-r} \exp(-c_{11}\sqrt{\log x}\,)$$

Lemma 3 follows from Lemma 2 by partial summation. Similarly if we take χ principal in Lemma 1, then we obtain the following by partial summation.

Lemma 4. If $x \geq \exp(c_{10}\log^2 q)$ and $r \geq 2$, then

$$\sum_{\substack{n > x \\ (n,q)=1}} \mu(n)n^{-r} \ll x^{1-r} \exp(c_{11}\sqrt{\log x}\,)$$

3. PROOF OF THEOREM 1.

The methods used by Walfisz and Siebert bear a resemblance to the "hyperbola method" used in the Dirichlet divisor problem. If $y \leq x^{1/r}$, then (1) yields

$$S_r(x;q,a) = \sum_{\substack{d^r m \leq x \\ d^r m \equiv a \pmod q}} \mu(d)$$

$$\sum_{d \leq y} + \sum_{d > y}$$

$$= \sum\nolimits_1$$

say. Then

$$\sum\nolimits_1 \quad \sum_{d \leqslant y} \mu(d) \sum_{\substack{m \leqslant xd^{-r} \\ d^r m \equiv a (\mathrm{mod}\ q)}} 1$$

and clearly

$$\sum_{\substack{m \leqslant xd^{-r} \\ d^r m \equiv a (\mathrm{mod}\ q)}} 1 = \begin{cases} 0 & (d^r,q) \nmid a \\ \dfrac{x(d^r,q)}{qd^r} + O(1) , & (d^r,q) \mid a \end{cases}$$

Hence

$$\sum\nolimits_1 = \frac{x}{q} f(a,q) - \frac{x}{q} \sum_{\substack{\ell=1 \\ (\ell^r,q) \mid a}}^{q} (\ell^r,q) \sum_{\substack{d > y \\ d \equiv \ell (\mathrm{mod}\ q)}} \frac{\mu(d)}{d^r} + O(y)$$

If $y \geqslant \exp(c_{10} \log^2 q)$, then Lemma 3 yields

$$\sum\nolimits_1 = \frac{x}{q} f(a,q) + O(xy^{1-r} \exp(-c_{11} \sqrt{\log y})) + O(y) \tag{7}$$

From Lemma 2 it follows directly that

$$\sum\nolimits_2 = \sum_{m \leqslant xy^{-r}} \sum_{\substack{\ell=1 \\ \ell^r m \equiv a (\mathrm{mod}\ q)}}^{q} \{M((x/m)^{1/r};q,\ell) - M(y;q,\ell)\}$$

$$\ll x^{1/r} q \exp(-c_{11} \sqrt{\log y}) \sum_{m \leqslant xy^{-1}} m^{-1/r}$$

$$+ xqy^{1-r} \exp(-c_{11} \sqrt{\log y})$$

$$\leqslant xqy^{1-r} \exp(-c_{11}\sqrt{\log y}) .$$

We then choose $y = (xq)^{1/r} \exp(-\frac{c_{11}}{\sqrt{2}} r^{-3/2}\sqrt{\log x})$. Note that

$$\log y \geqslant r^{-1} \log x - c_{11}r^{-3/2}\sqrt{\log x}$$

$$\geqslant \frac{1}{2r} \log x$$

$$\geqslant c_{10} \log^2 q$$

if x is sufficiently large and $c_3 \geqslant 2c_{10}$. Finally,

$$\log y = \frac{1}{r} \log x + \frac{1}{r} \log q - \frac{c_{11}}{\sqrt{2}} r^{-3/2}\sqrt{\log x}$$

$$\leqslant \frac{1}{r} \log x + \frac{1}{r} \log q - c_{11}\sqrt{\frac{c_3}{2}}\, r^{-1} \log q$$

$$\leqslant \frac{1}{r} \log x$$

if $c_3 \geqslant 2c_{11}^{-2}$, so that $y \leqslant x^{1/r}$

4. PROOF OF THEOREM 2.

We use (6) again but estimate the sum $\sum_2$ in a different way. If $(a,q) = 1$, then

$$(8) \qquad \sum_2 = \sum_{\substack{md^r \leqslant x \\ md^r \equiv a \pmod q \\ d > y}} \mu(d)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \sum_{\substack{md^r \leqslant x \\ d > y}} \mu(d)\chi^r(d)\chi(m)$$

By Lemma 1 the inner sum satisfies

$$\sum_{m\leqslant xy^{-r}} \chi(m) \left\{ \sum_{d\leqslant (x/m)^{1/r}} \mu(d)\chi^r(d) - \sum_{d\leqslant y} \mu(d)\chi^r(d) \right\}$$

$$\ll x^{1/r} \exp(-c_{11}\sqrt{\log y}) \sum_{m\leqslant xy^{-r}} m^{-1/r} + xy^{1-r} \exp(-c_{11}\sqrt{\log y})$$

$$\ll xy^{1-r} \exp(-c_{11}\sqrt{\log y})$$

This time we choose $y = x^{1/r} \exp(-\frac{c_{11}}{\sqrt{2}} r^{-3/2}\sqrt{\log x})$, and the result follows from (6), (7), and (8).

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824

Current Address:
Department of Mathematics
DRB 306, University Park
University of Southern California
Los Angeles, California 90089-1113